3.0 LONG-PERIOD MOTION OF THE ORBIT PLANE OF A NATURAL SATELLITE

The orbital motion of most major natural satellites in the Solar System is characterised by small perturbations from two or more sources. These may be listed :

- The gravitational attraction of other satellites orbiting the same primary
- The gravitational attraction of the Sun
- The oblateness of the primary, which causes the gravitational potential field of the primary to differ from the simple $R = -GM/r$ of a spherically symmetric body or point mass

The relative magnitudes of these effects depends in part upon the orbital distance of the satellite relative to the radius of the primary and to the orbital distance of the Sun. A satellite orbiting within a few planetary radii will be strongly affected by the oblateness of the primary ; conversely, a satellite in a large orbit, whose period is of the order of a year or more, will suffer significant solar perturbations. In either case, the motion of the satellite may also be disturbed by the presence of another massive satellite in a nearby orbit.

The secular motion of the orbit plane is of interest because it is rather sensitive to the relative sizes of the various perturbing effects. In this section we consider the secular motion of the orbit plane of a satellite which is subject to perturbations of similar magnitude from two or more sources. As an example, we consider the case of Iapetus, the ninth satellite of Saturn, which is perturbed principally by Titan and the Sun. Oblateness perturbations upon Iapetus are smaller but are not negligible.

3.1 THE DISTURBING FUNCTION

We present below the terms from the disturbing function which contain only the node and inclination of the satellite and the perturbing object (or the equator plane of the primary in the case of oblateness perturbations). Terms are given to fourth order in the sine of the inclinations and they are with respect to an arbitrary fixed reference plane. As can be seen, the three disturbing functions are very similar in form.

Solar disturbing function

$$R_s^{(2)} = -n^2a^2\chi_s \{\sin^2 I + \sin^2 I_s - 2 \sin I \cos I \sin I_s \cos I_s \cos(\Omega - \Omega_s) \}$$

[1]

$$R_s^{(4)} = -n^2a^2\chi_s \{-\tfrac{1}{2} \sin^2 I \sin^2 I_s \cos 2(\Omega - \Omega_s) - 3/2 \sin^2 I \sin^2 I_s \}$$

Oblateness disturbing function

$$R_e^{\langle 2 \rangle} = -n^2 a^2 \chi_e \{\sin^2 I + \sin^2 I_e - 2 \sin I \cos I \sin I_e \cos I_e \cos(\Omega - \Omega_e)\}$$

[2]

$$R_e^{\langle 4 \rangle} = -n^2 a^2 \chi_e \{-\tfrac{1}{2} \sin^2 I \sin^2 I_e \cos 2(\Omega - \Omega_e) - 3/2 \sin^2 I \sin^2 I_e\}$$

Satellite disturbing function

$$R_t^{\langle 2 \rangle} = -n^2 a^2 \chi_t \{3 - \cos I - \cos I_t - \cos I \cos I_t - 2 \sin I \sin I_t \cos(\Omega - \Omega_t)\}$$

[3]

$$R_t^{\langle 4 \rangle} = -n^2 a^2 \chi_t \{-2 (1 - \cos I)(1 - \cos I_t) \cos 2(\Omega - \Omega_t)\}$$

Where I = Inclination of the satellite orbit to the reference plane

Ω = Longitude of the node of the satellite orbit upon the reference plane

$R^{\langle 2 \rangle}$ = first-order part of the disturbing function

$R^{\langle 4 \rangle}$ = second-order part of the disturbing function

χ_s = $(3/8)\ \mu_s\ (a/a_s)^3$

χ_e = $(3/4)\ J_2\ (a_e/a)^2$

χ_t = $(1/8)\ \mu_t\ \alpha b_{3/2}^{\langle 1 \rangle}$

and

M = mass of the primary

a = semi-major axis of the satellite orbit

n = mean motion of the satellite

J_2 = dynamical form factor of the primary

μ_j = the mass ratio of the disturbing body to the primary

α, $b_{3/2}^{(1)}$ = the ratio of the semi-major axis of the disturbing satellite to the disturbed satellite, and a Laplace coefficient.

Ω_t, I_t = Node and inclination of the orbit of the disturbing satellite on the reference plane

a_e = equatorial radius of the primary

Ω_e, I_e = Node and inclination of the equator plane of the primary on the reference plane

a_s = semi-major axis of the orbit of the Sun around the primary

Ω_s, I_s = Node and inclination of the orbit of the Sun on the reference plane.

The perturbation in the node and inclination may be found from the Lagrange planetary equations. We need only retain the terms involving partial derivatives of R with respect to Ω and i since the other osculating elements of the satellite do not appear in the disturbing functions as given in [1], [2], [3]. Thus

[4] $$\frac{d\Omega}{dt} = \frac{1}{na^2 \sin i} \frac{\partial R}{\partial i}$$

[5] $$\frac{di}{dt} = \frac{-1}{na^2 \sin i} \frac{\partial R}{\partial \Omega}.$$

3.2 FIRST-ORDER THEORY FOR TWO DISTURBING FORCES

If there are only two perturbing effects and we neglect powers of sine inclination above the second, then the disturbing function is of the following form.

[6] $$R = -n^2a^2\,\chi_1\,\{I^2 + I_1{}^2 - 2\,I\,I_1\cos(\Omega - \Omega_1)\}$$
$$-n^2a^2\,\chi_2\,\{I^2 + I_2{}^2 - 2\,I\,I_2\cos(\Omega - \Omega_2)\}$$

Consider the perturbation in the inclination.

[7] $$dI/dt = 2n\,\{\,\chi_1\,I_1\sin(\Omega - \Omega_1) + \chi_2\,I_2\sin(\Omega - \Omega_2)\,\}$$

We may choose the reference plane so that it passes through the points of intersection of the orbit planes of the disturbing bodies (or the equator plane of the primary if oblateness is one of the disturbing effects).

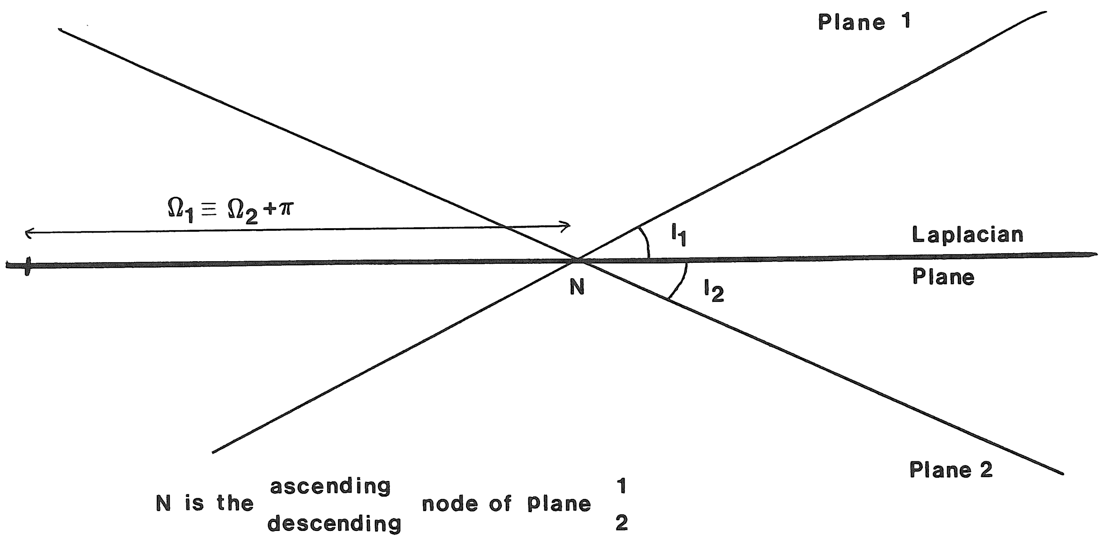

Figure 5. Laplacian plane

Then

[8] $\Omega_2 \equiv \Omega_1 - \pi$

and hence

[9] $\sin(\Omega - \Omega_2) \equiv -\sin(\Omega - \Omega_1)$

and we may write

[10] $dI/dt = 2n(\chi_1 I_1 - \chi_2 I_2)\sin(\Omega - \Omega_1).$

Clearly we may make dI/dt vanish if

[11] $$\chi_1 I_1 - \chi_2 I_2 \equiv 0.$$

This means that the inclination of the orbit of the satellite remains constant upon the plane defined in the figure. The plane is called the Laplacian plane of the satellite and it lies between the orbit planes of the disturbing bodies. Its inclination with respect to either of these planes depends upon the relative sizes of the disturbing forces, and may be found be solving [11] in conjunction with

[12] $$I_1 + I_2 = I^*$$

where I^* is the mutual inclination of the orbit planes of the disturbing bodies.

If $\underline{n}_1$ and $\underline{n}_2$ are the unit normal vectors to the orbit planes 1 and 2 respectively then the unit normal to the Laplacian plane is given by

[13] $$\underline{n}_L = (\sin I_2/\sin I^*)\ \underline{n}_1 + (\sin I_1/\sin I^*)\ \underline{n}_2$$

We may determine the motion of the node upon the Laplacian plane using [4].

[14] $d\Omega/dt = \kappa = -2n\,(\chi_1 + \chi_2)$

That is to say, the orbit precesses uniformly in a retrograde direction upon the Laplacian plane.

Let the node and inclination of the Laplacian plane upon a fixed ecliptic and equinox of epoch be Ω_L, I_L and suppose the orbit of the satellite to be inclined at a constant angle I' to the Laplacian plane. Moreover, denote by Ψ the arc of the Laplacian plane from its ascending node upon the ecliptic to the ascending node of the satellite orbit upon it, as in the accompanying figure.

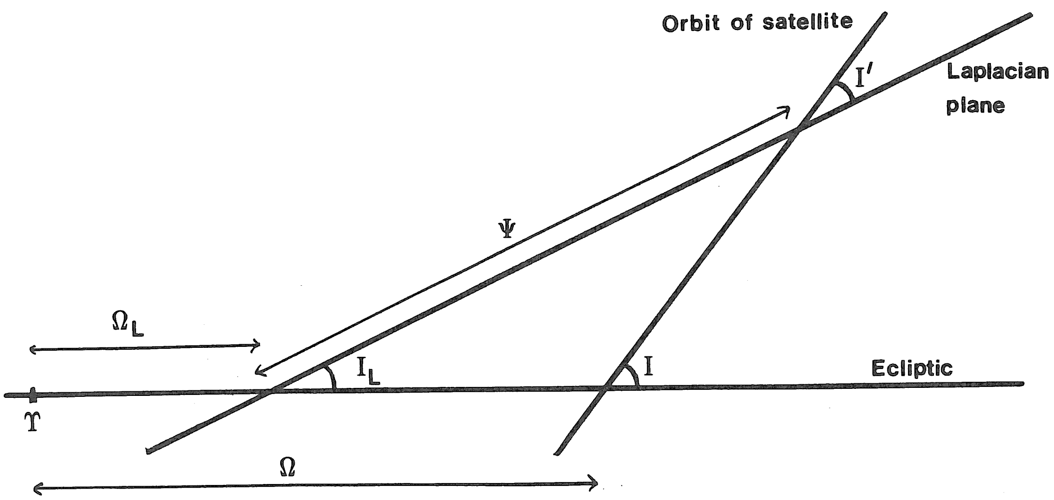

Figure 6. The Laplacian plane and the satellite orbit

Then the node and inclination of the satellite upon the ecliptic and equinox are given by

$$\begin{aligned} \sin(\Omega - \Omega_L)\sin I &= \sin I' \sin \Psi \\ \cos(\Omega - \Omega_L)\sin I &= \cos I' \sin I_L + \sin I' \cos I_L \cos \Psi \quad [15] \\ \cos I &= \cos I' \cos I_L - \sin I' \sin I_L \cos \Psi \end{aligned}$$

where

$$\Psi = \Psi_o - \kappa t.$$

Example : Iapetus

The principal perturbing forces upon Iapetus are due to the Sun and Titan. The action of oblateness and the other inner satellites is small and is included in the disturbing function due to Titan : the equator plane of Saturn is assumed to be identical to the orbit plane of Titan. We may write

$$\chi_s = (3/8)\ \mu_s\ (a/a_s)^3$$

$$\chi_t = (1/8)\ \mu_t\ \alpha b_{3/2}^{(1)} + (3/4)\ J_2\ (a_e/a)^2 + (1/8) \sum_i \mu_i \alpha_i b_{3/2}^{(1)}(\alpha_i)$$

and we adopt the following values for the parameters of the orbit of Iapetus, the Sun and Titan, and of J_2 of Saturn, referred to the mean ecliptic and equinox of 1950.

$\Omega_s = 113^\circ.158$ $\qquad I_s = 2^\circ.4909$

$\Omega_t = 168^\circ.747$ $\qquad I_t = 27^\circ.779$

$\mu_s = 3499.4$ $\qquad a/a_s = 0.0024948$

$\mu_t = 2.383\ 10^{-4}$ $\qquad \alpha = a_t/a = 0.34314$

$3/2\ J_2 = 0.024311$ $\qquad a_e/a = 0.016853$

$n = 4^\circ.53795711$ per day

Hence $\quad I^* = 26^\circ.445$

$\chi_s = 2.037\ 10^{-5}$

$\chi_t = 1.330\ 10^{-5} + 0.345\ 10^{-5} + 0.011\ 10^{-5} = 1.686\ 10^{-5}$

where the contribution of the oblateness and the other inner satellites to χ_t have been given explicitly (the second and third terms respectively) to show their relative sizes.

The inclination of the orbits of the Sun and Titan to the Laplacian plane may be determined to be

$$I_s = I^* \chi_t/(\chi_s + \chi_t) = 11^\circ.933$$

$$I_t = I^* \chi_s/(\chi_s + \chi_t) = 14^\circ.512.$$

The normal to the Laplacian plane is given by

$$\underline{n}_L = F_s \ \underline{n}_s + F_t \ \underline{n}_t$$

where $$F_s = \sin I_t/\sin I^*$$

$$F_t = \sin I_s/\sin I^*.$$

The components of $\underline{n}_L$ are thus (0.06471065, 0.22184702, 0.97293132) and hence the position of the pole of the Laplacian plane is

$$\Omega_L = 163^\circ.738$$

$$I_L = 13^\circ.3614$$

and the rate of motion of the orbit of Iapetus on its Laplacian plane is

$$\kappa = -11^\circ.3478 \text{ per Julian century.}$$

We may determine the position of the orbit of Iapetus upon its Laplacian plane by comparison with mean elements given by Struve (1933) and Sinclair (1974). The mean node and inclination at several epochs are presented in the table below, referred to the mean ecliptic and equinox of 1950. The values obtained directly from observations include long period perturbations due to the Sun. These terms must be subtracted in order to determine the underlying secular variations of the node and inclination. Struve subtracted one Solar term in forming his mean points but the theory of the Solar perturbations upon Iapetus has been revised by Sinclair (1974) and by Harper et al (in submission). The points from Struve have been corrected by subtraction of the periodic perturbations given by Sinclair (1974) and Harper et al (in submission) ; the point from Sinclair (1974) has been corrected by subtracting the periodic perturbation in Harper et al.

Date	Node	Inclination
1787.70	146°.50627	19°.26139
1832.50	145°.05501	18°.85991
1857.50	144°.32894	18°.70378
1876.70	143°.39184	18°.54219
1880.20	143°.35909	18°.54550
1885.60	143°.10195	18°.46443
1917.20	141°.96303	18°.15884
1918.20	141°.91871	18°.15050
1926.40	141°.55608	18°.06129
1927.40	141°.57161	18°.05516
1973.00	139°.89829	17°.56719

We solve for Ω and I in equations [15] and for μ_T. The values obtained are

$$\Omega = 143°.084 \pm 0°.040 \text{ at the epoch } 1885.25$$

$$I = 18°.449 \pm 0°.013$$

$$\mu_T = (2.333 \pm 0.063)\times 10^{-4}.$$

The residuals $\sin \iota \, \Delta\Omega$ and $\Delta\iota$ are given in the following table.

Date	$\sin \iota \; \Delta\Omega$	$\Delta\iota$
1787.70	$-0^\circ.0997$	$-0^\circ.0139$
1832.50	$-0^\circ.0096$	$-0^\circ.0576$
1857.50	$+0^\circ.0611$	$+0^\circ.0003$
1876.70	$-0^\circ.0089$	$+0^\circ.0100$
1880.20	$+0^\circ.0220$	$+0^\circ.0452$
1885.60	$+0^\circ.0039$	$+0^\circ.0136$
1917.20	$+0^\circ.0102$	$+0^\circ.0071$
1918.20	$+0^\circ.0077$	$+0^\circ.0085$
1926.40	$-0^\circ.0128$	$-0^\circ.0004$
1927.40	$+0^\circ.0032$	$+0^\circ.0033$
1973.00	$-0^\circ.0134$	$-0^\circ.0192$

Root-mean-square	$0^\circ.0368$	$0^\circ.0240$

Inclination of the orbit of Iapetus to its Laplacian plane = $7^\circ\ 33'.0$

3.3 FIRST-ORDER THEORY WITH MORE THAN TWO DISTURBING FORCES

When there are more than two significant disturbing forces, the formulation of the previous section is less tractable. Equation [7] has three or more terms on the right-hand side and cannot be readily transformed into the form of equation [10]. We elect to use a different approach by introducing new variables p and q defined by

$$p = \sin I \sin \Omega$$

[16]

$$q = \sin I \cos \Omega.$$

Lagrange's planetary equations now become

$$na^2\, dp/dt = + \cos I\, \partial R/\partial q \simeq + \partial R/\partial q$$

[17]

$$na^2\, dq/dt = - \cos I\, \partial R/\partial p \simeq - \partial R/\partial p.$$

This form of the equations was used by Tisserand (1892). We use the approximate forms, assuming $\cos I \simeq 1$.

The first-order disturbing functions $R_s^{\langle 2 \rangle}$, $R_t^{\langle 2 \rangle}$, $R_e^{\langle 2 \rangle}$ may each be written in the form

[18] $$R_i^{\langle 2 \rangle} = -n^2a^2\, \chi_i\, \{p^2 + q^2 - 2\alpha_i p - 2\beta_i q + \alpha_i^2 + \beta_i^2\}.$$

The total disturbing function due to N perturbing forces may be written

[19] $$R = -n^2a^2K\, \{p^2 + q^2 - 2Ap - 2Bq + C\}$$

where

$$K = \sum_i \chi_i$$

[20]

$$A = (\sum_i \chi_i \alpha_i)/K$$

$$B = (\sum_i \chi_i \beta_i)/K$$

$$C = (\sum_i \chi_i \, (\alpha_i^2 + \beta_i^2) \,)/K.$$

Tisserand (1892) showed that the disturbing function has the following property when restricted to the secular terms described in this work.

[21] $$\frac{dR}{dt} = \frac{dp}{dt}\frac{\partial R}{\partial p} + \frac{dq}{dt}\frac{\partial R}{\partial q} = 0$$

That is to say

[22] $$R = \text{constant}$$

hence

[23] $$p^2 + q^2 - 2Ap - 2Bq + C = \text{constant.}$$

This is the equation of a circle

[24] $$(p - p_o)^2 + (q - q_o)^2 = r^2$$

where

[25] $$p_o = A \quad , \quad q_o = B$$

define the centre of the circle in the pq-plane. Referring back to the previous section, it is evident that (p_o, q_o) is the position of the pole of the Laplacian plane of the orbit. The orbit maintains a constant inclination to the Laplacian plane given by $r = \sin I$ where r^2 is the constant right-hand side of equation [24]. If Ω_L, I_L denote the node and inclination of the Laplacian plane in the fixed reference system then

[26] $$p_o = \sin I_L \sin \Omega_L$$
$$q_o = \sin I_L \cos \Omega_L.$$

We may regard p_o, q_o as the coordinates of the 'centroid' or 'weighted mean' of the coordinates of the poles of the disturbing forces. The 'weight' of each pole (α_i, β_i) is the dimensionless coefficient χ_i in equation [18].

If we choose the coordinate system so that the reference plane is the Laplacian plane then $I_L = 0$ and hence $p_o = q_o = 0$. We then have

[27] $$R = -n^2a^2 K (p^2 + q^2 + C')$$

$$= -n^2a^2 K U.$$

Applying equations [17] we have

$$dp/dt = - nK \partial U/\partial q = -2n K q$$

[28]

$$dq/dt = + nK \partial U/\partial p = +2n K p.$$

The solution to these equations may be written

$$p = r \sin (\kappa t - \phi)$$

[29]

$$q = r \cos (\kappa t - \phi)$$

where

[30] $$\kappa = -2n K$$

and r, ϕ are to be determined from observations. ϕ is the node of the orbit plane upon the Laplacian plane at time t = 0 and r is as in equation [24].

The rate of precession of the orbit upon the Laplacian plane is 2K times the mean motion of the satellite - cf. equation [14].

3.4 SECOND-ORDER THEORY WITH AN ARBITRARY NUMBER OF DISTURBING FORCES

We now consider the disturbing functions [1], [2], [3] up to fourth order in the inclinations, i.e. including terms such as $R_s^{\langle 4 \rangle}$, $R_t^{\langle 4 \rangle}$, $R_e^{\langle 4 \rangle}$. Using the notation of the previous section together with

$$D = \left(\sum_i \chi_i \, \alpha_i \, \beta_i \right)/K$$

[31] $$\gamma = \left(\sum_i \chi_i \, k_i \, (\alpha_i^2 + \beta_i^2) \right)/K$$

$$\varepsilon = \tfrac{1}{2} \left(\sum_i \chi_i \, (\beta_i^2 - \alpha_i^2) \right)/K$$

we have

[32] $$R = - n^2 a^2 K \{(1 - \gamma + \varepsilon)p^2 + (1 - \gamma - \varepsilon)q^2 - 2Ap - 2Bq - 2Dpq + C\}.$$

This differs from equation [19] in two respects : (i) the coefficients of p^2 and q^2 are no longer equal, and (ii) there is now a second-order cross term in pq. Equations [22] and [32] imply that the pole of the orbit follows an ellipse. The centre of the ellipse (p_o, q_o) may be found by substituting $p = p_o + x$, $q = q_o + y$ into [32] and making the resulting coefficients of x and y equal to zero. We find

$$p_o = (BD + A(1 - \gamma - \varepsilon))/\Gamma^2$$

[33]

$$q_o = (AD + B(1 - \gamma + \varepsilon))/\Gamma^2$$

where

$$\Gamma^2 = (1 - \gamma)^2 - \varepsilon^2 - D^2.$$

This is the pole of the Laplacian plane of the orbit - cf. equation 25. The disturbing function becomes

[34] $$R = - n^2a^2K\{ (1 - \gamma - \varepsilon)x^2 + (1 - \gamma + \varepsilon)y^2 - 2Dxy + G\}$$

where G is a new constant. Then

[35]
$$dp/dt = dx/dt = - 2nK \{ (1 - \gamma + \varepsilon)y - Dx\}$$
$$dq/dt = dy/dt = + 2nK \{ (1 - \gamma - \varepsilon)x - Dy\}.$$

These equations admit the general solution

[36]
$$x = x_c \cos \kappa t + x_s \sin \kappa t$$
$$y = y_c \cos \kappa t + y_s \sin \kappa t$$

where the rate of precession κ is given by

[37] $$\kappa = - 2nk \Gamma.$$

The four coefficients x_c, y_c, x_s, y_s are not independent. If we choose x_c, y_c to be arbitrary constants determined from observations then x_s, y_s are

[38]
$$x_s = \{+ Dx_c - (1 - \gamma - \varepsilon)y_c\}/\Gamma$$
$$y_s = \{- Dy_c - (1 - \gamma + \varepsilon)x_c\}/\Gamma .$$

Example : Iapetus

We may refine the theory of the motion of the orbit plane of Iapetus by re-calculating the rate of precession λ and the position of the pole of the Laplacian plane. Using the previous position of the Laplacian plane as a reference plane we may define a triad of orthonormal vectors $\underline{N}$, $\underline{M}$, $\underline{W}$ such that $\underline{N}$ is in the line of intersection of the Laplacian plane with the ecliptic, at the ascending node; $\underline{M}$ is in the Laplacian plane 90° from $\underline{N}$ and $\underline{W}$ is normal to the Laplacian plane.

Let $\Omega_L{}^0$, $I_L{}^0$ be the node and inclination of the Laplacian plane obtained in the previous section and now to be used as a first approximation. Then

$$\underline{N}_L{}^0 = \begin{cases} + \cos \Omega_L{}^0 \\ + \sin \Omega_L{}^0 \\ 0 \end{cases}$$

$$\underline{M}_L{}^0 = \begin{cases} - \sin \Omega_L{}^0 \cos I_L{}^0 \\ + \cos \Omega_L{}^0 \cos I_L{}^0 \\ + \sin I_L{}^0 \end{cases}$$

$$\underline{W}_L{}^0 = \begin{cases} + \sin \Omega_L{}^0 \sin I_L{}^0 \\ - \cos \Omega_L{}^0 \sin I_L{}^0 \\ + \cos I_L{}^0 \end{cases}$$

Let Ω_i, I_i be the node and inclination of the orbit of the disturbing body referred to the ecliptic and equinox of 1950 and $\underline{n}_i$ be the unit normal to that plane. Then

$$\alpha_i = \underline{n}_i \bullet \underline{N}_L{}^0$$

$$\beta_i = \underline{n}_i \bullet \underline{M}_L{}^0$$

We tabulate below the values of α, β and χ for the Sun, Titan and the oblateness of Saturn. We adopt the following parameters (cf. previous example.)

Sun : $\Omega_s = 113^\circ.158$ $I_s = 2^\circ.4909$

$\chi_s = 2.03726 \ 10^{-5}$

Oblateness : $\Omega_e = 168°.710$ $I_e = 28°.1410$

$\chi_e = 0.34525\ 10^{-5}$

Titan : $\Omega_t = 168°.747$ $I_t = 27°.7790$

$\chi_t/\mu_t = 0.0558215$

$\mu_t = 2.3829\ 10^{-4}$ $\chi_t = 1.33017\ 10^{-5}$

Mean motion of Iapetus = 4°.53795711 per day

First-order position of the Laplacian plane :

$\Omega_L{}^0 = 163°.723$ $I_L{}^0 = 13°.3670$

Body	α	β	$\chi \times 10^5$
Sun	-0.033491	+0.199849	2.0373
Titan	+0.042086	-0.251164	1.3217
Oblateness	+0.042287	-0.257276	0.3452

We obtain the following results upon fitting to the data.

Rate of precession of the orbit upon the Laplacian plane :

$\kappa = -\ 12°.2793$ per Julian century

(cf. -11°.3478 in the first-order theory)

Position of the pole of the Laplacian plane :

$$p_o = +0.0005837$$

$$q_o = -0.0039601$$

$$\Omega_L = 163^\circ.711$$

$$I_L = 13^\circ.3497$$

Solution of the equations for p and q :

$$p = p_o \quad -0.111288 \cos \kappa t \quad +0.068650 \sin \kappa t$$

$$q = q_o \quad -0.069426 \cos \kappa t \quad -0.114809 \sin \kappa t$$

We may deduce the maximum and minimum values of the inclination of the orbit of Iapetus to the Laplacian plane :

$$I_{maximum} = 7^\circ\ 43'.0$$

$$I_{minimum} = 7^\circ\ 30'.5$$

The range of inclination is thus $7^\circ\ 36'.8 \pm 6'.3$ (cf. $7^\circ\ 33'.0$ in the first-order case)

The residuals $\sin \iota\ \Delta\Omega$ and $\Delta\iota$ are given in the following table.

Date	$\sin \iota \, \Delta\Omega$	$\Delta\iota$
1787.70	-0°.1627	-0°.0435
1832.50	-0°.0456	-0°.0744
1857.50	+0°.0406	-0°.0084
1876.70	-0°.0176	+0°.0078
1880.20	+0°.0155	+0°.0441
1885.60	+0°.0008	+0°.0145
1917.20	+0°.0270	+0°.0194
1918.20	+0°.0252	+0°.0212
1926.40	+0°.0100	+0°.0154
1927.40	+0°.0266	+0°.0195
1973.00	-0°.0391	-0°.0152

Root-mean-square   0°.0560   0°.0322

The values of the node and inclination of the orbit of Iapetus at the epoch 1885.25 and the mass of Titan are found to be :

$$\Omega = 143^{\circ}.093 \pm 0^{\circ}.045$$

$$I = 18^{\circ}.448 \pm 0^{\circ}.014$$

$$\mu_T = (2.252 \pm 0.067)\times 10^{-4}$$

3.5 DISCUSSION OF RESULTS : DETERMINATION OF THE MASS OF TITAN

Previous authors have used the motion of the orbit plane of Iapetus to determine the mass of Titan. As we have shown, the rate of precession of the orbit plane upon its Laplacian plane is directly dependent upon the mass of Titan (cf. equations [14], [30], [37] and the expansions for χ_s, χ_e, χ_t given after equation [3]).

We give below a table of values from Jeffreys (1953), Sinclair (1974) and this work, plus the values obtained by Sinclair and Taylor (1985) from an analysis of the orbits of Titan, Hyperion and Iapetus by numerical integration, by Tyler et al (1981) from analysis of Voyager 1 radio-tracking data and by Message from the motion of Hyperion where value (a) is a weighted mean of values obtained from individual terms in the theory of Hyperion and value (b) is a least-squares solution.

Source	$\mu_T \times 10^4$
Jeffreys	2.412 ± 0.018
Jeffreys (from Iapetus)	2.357 ± 0.052
Sinclair	2.422 ± 0.031
1^{st}-order Laplacian plane	2.333 ± 0.063
2^{nd}-order Laplacian plane	2.252 ± 0.067
Sinclair and Taylor	2.36777 ± 0.00055
Tyler et al	2.3664 ± 0.0008
Message (a)	2.3648 ± 0.0055
Message (b)	2.3677 ± 0.0004

These values may be visualised in the following figure.

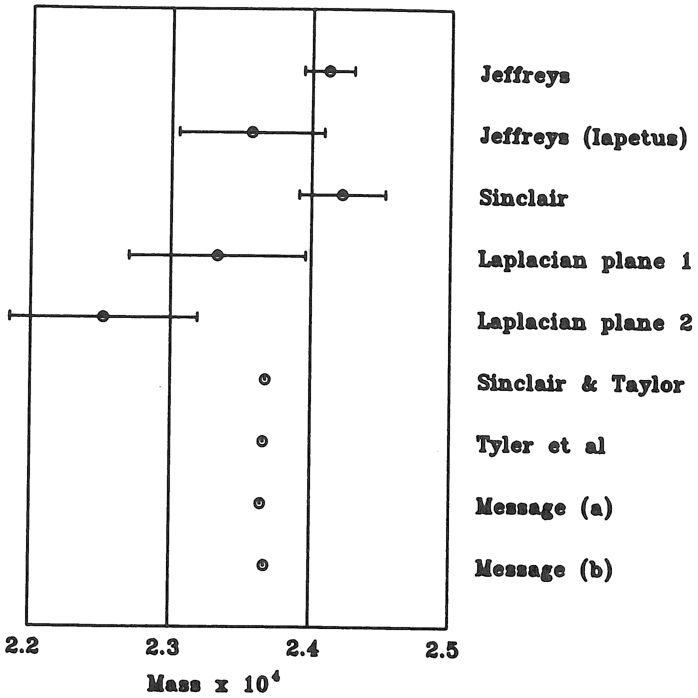

Figure 7. Determination of the mass of Titan

The values of μ_T obtained here are consistent with those obtained by other authors, notably Sinclair and Taylor, Tyler et al and Jeffreys' determination from the motion of the orbit plane of Iapetus. The value from the first-order theory is in better agreement with other values than that from the second-order theory. In addition, the first-order theory gives a better fit to the data since its root-mean-square residuals in $\sin \iota\ \Delta\Omega$ and $\Delta\iota$ are smaller by some 30% than those from the second-order theory. We are probably justified, therefore, in preferring the value of μ_T from the first-order theory as the more reliable.

The main difference between the two theories, particularly when fitting to data over a period of two centuries, lies in the precession rate κ. The second-order theory has a rate some 8% larger than the first-order theory due to the factor Γ and we may expect the two models to diverge over long periods of time. The difference amounts to $0^\circ.931$ per Julian century or $1^\circ.73$ over the span of the data. This will not be manifested in the calculated values of the node and inclination however, since the parameters of the model will change to accommodate different precession rates as part of the least-squares fitting process. It may be that the mass of Titan from the second-order theory is low because the fitting process tried to reduce the precession rate : the precession rate implied (in the second-order expression for κ) by a value of $\mu_T = 2.35\ 10^{-4}$ (say) may in fact be too large than the rate represented by the observed values of node and inclination. However, the data only covers 200 years of a 3000 year precession and the rate is not well-determined anyway, as the large standard error of μ_T attests. The mass of Titan may be obtained more accurately by other means.

It is important to note that the fourth-order terms in the disturbing functions given at the start of this chapter do not contain the fourth power of the inclination of the satellite. They are fourth-order only in the sense that the powers of the various inclinations in each term sum to 4. This allows us to retain the more tractable form of the equations for dp/dt and dq/dt. Inclusion of the fourth power of the satellite's inclination would introduce terms in p^3 and q^3 into the expressions for dq/dt and dp/dt respectively. In the case of mutual satellite terms we would also need to expand the planetary disturbing function to include terms whose Laplace coefficients have fractional subscript 3/2.